Int. J. Solids Structures Vol. 21, No. 6, pp. 527–543, 1985
Printed in the U.S.A.

0020-7683/85 \$3.00+.00

BOUNDING THEOREMS FOR CREEP-PLASTICITY

P. CARTER†
Department of Civil Engineering, University of Zimbabwe, Mount Pleasant, Harare, Zimbabwe

(*Received 12 December 1983; in revised form 12 August 1984*)

Abstract—Using an internal variable model, a number of structural theorems are proved that are of importance in high-temperature mechanical design. The theorems depend on convexity requirements that prohibit the conventional model of creep-plasticity. The application of reference stress methods is extended up to the limit load for steady loading and to the shakedown load for cyclic loading.

NOTATION

Q_i, q_i	load and deflection components
X_α, χ_α	internal force and internal variable components
$\bar{\chi}_\alpha$, χ_α^T	inelastic and thermal components of χ_α
ρ_α, ρ_α^T	internal residual force, thermal residual force
f	free (Helmholtz) or strain energy
h	Gibbs energy
j	complementary strain energy
e, p	superscripts denoting elastic and inelastic components of q_i, f, h, j
Ω	kinetic potential
$\phi(X_\alpha)$, $\Phi(Q_i)$	limit functions
k, n, ϕ_y	material constants
W, $\bar{W}$	functionals used in bounding theorems
U	maximum complementary work
ss, cp, sc, rc	superscripts denoting various histories of X_α, ρ_α, namely, steady state, cyclic plasticity, steady cyclic and rapid cyclic states
D	energy dissipation rate

1. INTRODUCTION

The reconciliation of thermodynamics with creep and plasticity has been an active area of research for >30 years. Two approaches have emerged. One is based on the assumption that the current state of the thermodynamic system is a functional of its history[1]. The other, used in this article, assumes that there exists a set of state variables, which at any instant defines the state of the system. Time-independent plasticity with kinematic strain hardening, creep and viscoplasticity may be modelled using the classical framework of irreversible thermodynamics with appropriate kinetic equations[2, 3]. This has been carried a stage further by Lambermont[4, 5], who identified the internal variables with dislocations, and Ponter *et al.*[6], who showed that their behaviour can be understood using a strain energy function with local instabilities. These successes with constitutive equations suggest the problem of obtaining structural theorems. Previous studies in this field have been concerned with minimum work and maximum complementary work functions[7] and the rate theorems of plasticity[8, 9].

Since 1963 a number of theorems have been obtained that are of great practical importance in high-temperature design and analysis[10–13]. Essentially they allow one to obtain upper bounds to work or displacement, or a "reference stress," using a stress analysis involving a relatively simple material. The results of Leckie[10] and Marriott[11] for a constant load and those of Ponter[12] for cyclic loading give bounds for an elastic-creep material in terms of the elastic solution plus an arbitrary constant residual stress. More generally, Ainsworth[13] used an elastic–plastic material to give

† Present address: NEI Cochrane Engineering (Pvt) Ltd., P.O. Box ST 361, Southerton Harare, Zimbabwe.

bounds for an elastic–plastic-creep material. These theorems may be used to obtain a qualitative idea of creep deformation in structures, which is often of more use to designers than the results of a full inelastic analysis. The purpose of this study is to prove and extend these theorems using the internal variable formalism of Kestin and Rice[2]. The arguments will be kept as general as possible, thereby demonstrating the essential properties on which the theorems depend. Considerable simplifications occur if the free (Helmholtz) energy (the strain energy) is a quadratic function, in which case it and the displacements may each be divided into elastic and inelastic components. This assumption is correct for metals. Further, there are convexity requirements on the energy dissipation rate function and the kinetic equations. This ensures stability in the time-independent case. Where possible, theorems will be obtained without assuming a particular form for the kinetic equations. They are then applied to the case of the creep-plasticity interaction, which is of particular practical interest. The convexity requirement prohibits the conventional model of creep-plasticity. An alternative is proposed that is less conservative (higher strain rates), but that is easier to handle theoretically in that there are no discontinuities.

The theorems of interest are concerned with minimum principles for steady and cyclic states and with displacement and work bounds for steady and cyclic loading. Steady and cyclic states may be defined for all load histories that do not violate the limit criterion.

Since the distinction between structure and material is to some extent arbitrary, the arguments will be presented in terms of generalised coordinates.

2. INTERNAL VARIABLE MODEL

2.1 *State variables and equations of state*

Consider an element subjected to macroscopic deflections q_i ($i = 1, 2, \ldots, n$), a set of internal variables χ_α and a set of internal temperatures T_α, where $\alpha = 1, 2, \ldots, \nu$. The internal variables are observable but are not, in general, directly controllable in the sense that they cannot be coupled directly to an external work force. (However, it is possible for a component of χ_α, say, a thermal component that is a function of T_α, to be independently controlled.) The thermodynamic state of the element is defined by the state variables (q_i, χ_α, T_α). The fundamental equation gives the free (Helmholtz) energy $f = f(q_i, \chi_\alpha, T_\alpha)$. The equations of state are $Q_i = \partial f/\partial q_i$, $X_\alpha = -\partial f/\partial \chi_\alpha$, $s_\alpha = -\partial f/\partial T_\alpha$, where Q_i is the applied force, X_α is the internal force and $S = \Sigma_\alpha s_\alpha$ is the entropy of the element. Throughout this paper, it is assumed that the only effect of temperature is on the thermal component of χ_α. Temperature may therefore be ignored as a state variable. Thermal effects will be referred to only in connection with cyclic loading.

If the element is an infinitesimal one, then q_i, Q_i are strain and stress components, respectively, f is the free energy per unit volume and S is the entropy per unit volume. The nature of the internal variables depends on what is regarded as an adequate description of the material. Physically they represent dislocations (see, for example, [3–6,14]), but may also be used to describe material behaviour in a phenomenological manner[15,16]. In the latter case, it is not clear that they would fit into this thermodynamically based framework, involving as it does extensive and conjugate intensive parameters related via a potential (energy) function. In this paper, we take the reductionist view that the important features of commercial high-temperature alloys, namely, plasticity with kinematic hardening up to a limit surface and primary and secondary creep with recovery, can be modelled using, for example, a two-element structure of the type shown in Fig. 1. The dashpot displacements are the internal variables, obeying kinetic equations of the form given in Section 2.2.

If the element is a finite element or structure, then the internal variables could be plastic strains, in which case the internal forces are the corresponding stresses. They could also represent cracks[17], but in this case the free energy and displacements

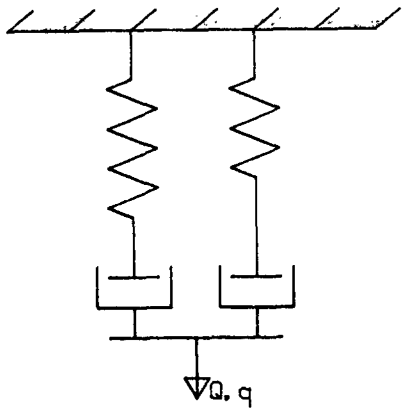

Fig. 1. Material model with springs and dashpots.

cannot be divided into elastic and inelastic parts since the free energy is not quadratic and convex.

If a different set of independent variables is required, say, (Q_i, χ_α), a Legendre transformation gives the new fundamental equation and equations of state[18]. In this case, the Gibbs potential is used: $h = Q_i q_i - f$. It is readily shown that $h = h(Q_i, \chi_\alpha)$ and that $q_i = \partial h/\partial Q_i$ and $X_\alpha = \partial h/\partial \chi_\alpha$ are the new equations of state. Carrying this further, we may define the complementary strain energy $j = h - X_\alpha \chi_\alpha$. Then $j = j(Q_i, X_\alpha)$ and $q_i = \partial j/\partial Q_i$, $\chi_\alpha = -\partial j/\partial X_\alpha$. Stability requirements ensure that these transformations can be performed.

In the case of a quadratic free energy function, the deflections and the free energy may be divided into elastic and inelastic components[2]. Thus,

$$f = f^e(q_i^e) + f^p(\chi_\alpha)$$

$$q_i = q_i^e + q_i^p(\chi_\alpha).$$

The Gibbs energy is then

$$h = h^e(Q_i) + Q_i q_i^p - f^p(\chi_\alpha),$$

where

$$h^e = Q_i q_i - f^e.$$

The corresponding equations of state are

$$q_i = \frac{\partial h}{\partial Q_i} = \frac{\partial h^e}{\partial Q_i} + q_i^p(\chi_\alpha) \tag{1a}$$

$$X_\alpha = \frac{\partial h}{\partial \chi_\alpha} = Q_i \frac{\partial q_i^p}{\partial \chi_\alpha} - \frac{\partial f^p}{\partial \chi_\alpha}. \tag{1b}$$

It is useful to define $\rho_\alpha = -\partial f^p/\partial \chi_\alpha$, which is a "residual" internal force. Then eqn (1b) becomes

$$X_\alpha = Q_i \frac{\partial q_i^p}{\partial \chi_\alpha} + \rho_\alpha. \tag{1c}$$

It may be shown that

$$j = h^e(Q_i) + j^p(\rho_\alpha), \tag{2a}$$

where

$$j^p = -\rho_\alpha \chi_\alpha - f^p \tag{2b}$$

and

$$\chi_\alpha = -\frac{\partial j^p}{\partial \rho_\alpha}. \tag{2c}$$

Since f is quadratic, it is easily shown that $f^e = h^e$ and $f^p = j^p$. The problem of determining f^e and f^p for a structure is left for a future paper.

2.2 *Kinetic equations and stability*

The description of the material is completed with a set of rate or kinetic equations giving the rates of change of the internal variables. Again, following Kestin and Rice[2], these will be assumed to be of the form

$$\dot{\chi}_\alpha = \frac{\partial \Omega}{\partial X_\alpha},$$

where $\Omega(X_\alpha)$ is termed the kinetic potential function. These equations imply the Onsager reciprocity relations. They may be inverted using a Legendre transformation provided $\partial^2\Omega/\partial X_\alpha \partial X_\beta$ is nonsingular.

The second part of the second law of thermodynamics requires the energy dissipation rate $D = X_\alpha \dot{\chi}_\alpha$ to be nonnegative. If the kinetic equations can be inverted, we may write $D = D(\dot{\chi}_\alpha)$ or $D = D(X_\alpha)$. To minimise the number of symbols, which function is intended will be made clear by writing its argument.

For time-dependent materials, Martin[19] showed that stability in the thermodynamic sense and in the sense of Drucker[20] was guaranteed if the free energy f and the energy dissipation rate $D(\dot{\chi}_\alpha)$ were positive convex functions of their respective arguments. For the case of time-independent plasticity, the convexity of $D(\dot{\chi}_\alpha)$ implies and is implied by the convexity of $\phi(X_\alpha)$, the plastic limit surface. For n-power creep, $\Omega = 1/(n+1)\cdot D(X_\alpha)$ and $D(\dot{\chi}_\alpha)$ are both convex. However, there is no physical reason why Ω should be proportional to D. Either a nonassociated flow rule or a nonhomogeneous kinetic potential would imply $\Omega \neq \text{constant}\cdot D(X_\alpha)$. In the following sections, theorems are proved in the general case, when it appears to be necessary to assume that both $\Omega(X_\alpha)$ and $D(\dot{\chi}_\alpha)$ are convex. The convexity of Ω is expressed by

$$\Omega(X''_\alpha) - \Omega(X'_\alpha) \geq \frac{\partial \Omega}{\partial X'_\alpha}(X''_\alpha - X'_\alpha). \tag{3a}$$

It follows that

$$(X''_\alpha - X'_\alpha)(\dot{\chi}''_\alpha - \dot{\chi}'_\alpha) \geq 0. \tag{3b}$$

An inequality similar to (3a) expresses the convexity of $D(\dot{\chi}_\alpha)$. The convexity of f implies the convexity of f^e, f^p, h^e and j^p.

One of the intentions in this study is to examine the creep-plasticity interaction. The usual model (see, for example, [12] or [13]) consists of n-power creep for stresses $\sigma \leq \sigma_y$, and perfect or hardening plasticity governed by the yield stress σ_y. The dashed line in Fig. 2 illustrates this law, and Fig. 3 shows the energy dissipation rate $D(\dot{\epsilon}^p)$. It is easy to show that $D(\dot{\epsilon}^p)$ is convex, provided $\sigma \leq (n/n+1)\sigma_y$. Although there is no *a priori* physical reason why $D(\dot{\epsilon}^p)$ or $D(\dot{\chi}_\alpha)$ should be convex, there is, it may be argued, an intuitive one, and it is of interest to construct a rate equation exhibiting n-power creep with plasticity as a limiting case and that has a convex $D(\dot{\chi}_\alpha)$. It appears that only one such curve is possible if there are to be no discontinuities. This is shown

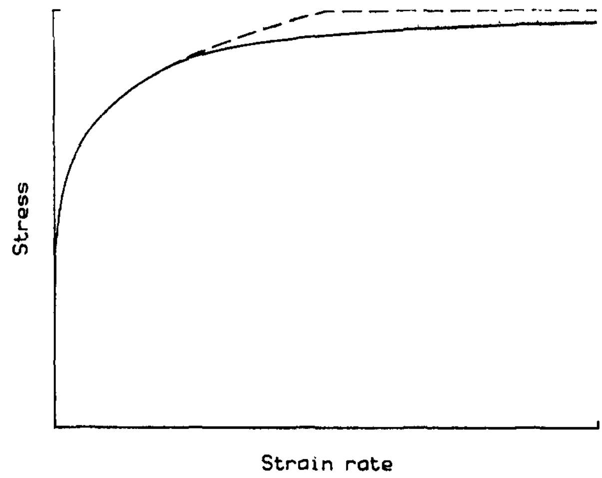

Fig. 2. Creep-plasticity kinetic relations ($n = 6$).

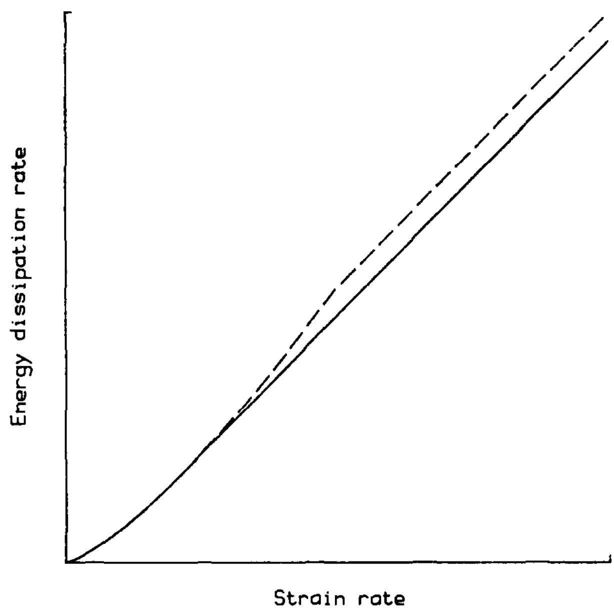

Fig. 3. Dissipation rates for creep-plasticity ($n = 3$).

as the solid line in Figs. 2 and 3, and may be represented as follows. Let $\phi(X_\alpha)$ be a homogeneous convex function of degree 1 in the components X_α. Then

$$\dot{\chi}_\alpha = \begin{cases} k\phi^n \dfrac{\partial\phi}{\partial X_\alpha}, & \text{if } \phi \leq \left(\dfrac{n}{n+1}\right)\phi_y \\ \dfrac{k/n\{[n/(n+1)]\phi_y\}^{n+1}\partial\phi/\partial X_\alpha}{\phi_y - \phi(X_\alpha)}, & \text{if } \left(\dfrac{n}{n+1}\right)\phi_y \leq \phi \leq \phi_y, \end{cases} \tag{4}$$

where k, n and ϕ_y are material constants (but may be temperature dependent). Then

$$D(X_\alpha) = \begin{cases} k\phi^{n+1}(X_\alpha), & \text{if } \phi \leq \left(\dfrac{n}{n+1}\right)\phi_y \\ \dfrac{k/n\{[n/(n+1)]\phi_y\}^{n+1}\phi(X_\alpha)}{\phi_y - \phi(X_\alpha)}, & \text{if } \left(\dfrac{n}{n+1}\right)\phi_y \leq \phi \leq \phi_y. \end{cases} \tag{5}$$

It can be shown that $\dot{\chi}_\alpha = \partial\Omega/\partial X_\alpha$ where

$$\Omega = \begin{cases} \frac{1}{n+1} D(X_\alpha), & \text{if } \phi \leq \left(\frac{n}{n+1}\right) \phi_y \\ k\{[n/(n+1)]\phi_y\}^{n+1} \left\{\frac{1}{n} \log_e \frac{\phi_y}{n+1[\phi_y - \phi(X_\alpha)]} + \frac{1}{n+1}\right\}, & \text{if } \left(\frac{n}{n+1}\right) \phi_y \leq \phi \leq \phi_y. \end{cases}$$

Also

$$\frac{\partial D}{\partial \dot{\chi}_\alpha} = \begin{cases} \frac{n+1}{n} X_\alpha, & \text{if } \phi \leq \left(\frac{n}{n+1}\right) \phi_y \\ \frac{\phi_y}{\phi(X_\alpha)} X_\alpha, & \text{if } \left(\frac{n}{n+1}\right) \phi_y \leq \phi \leq \phi_y. \end{cases} \tag{6}$$

If $\phi(X_\alpha) = \phi_y$, then the internal forces X_α are on the limit surface ϕ_y. Since $X_\alpha = Q_i\partial q_i^p/\partial\chi_\alpha + \rho_\alpha$ [eqn (1c)], we may write $\phi = \phi(Q_i, \rho_\alpha)$. Using the lower bound theorem of plasticity, it can be shown that the limit surface in Q_i − space is given by $\Phi(Q_i) = \phi_y$, where

$$\Phi(Q_i) = \phi(Q_i, \bar{\rho}_\alpha) \leq \phi(Q_i, \rho_\alpha) \tag{7}$$

for any ρ_α.

In the case where χ_α has a thermal component χ_α^T, we write $\dot{\chi}_\alpha = \dot{\bar{\chi}}_\alpha + \dot{\chi}_\alpha^T$, where $\dot{\bar{\chi}}_\alpha = \partial\Omega/\partial X_\alpha$ as shown above. Also $D = X_\alpha\dot{\bar{\chi}}_\alpha$.

3. EQUILIBRIUM, COMPATIBILITY AND VIRTUAL WORK

Equation (1c), namely

$$X_\alpha = Q_i \frac{\partial q_i^p}{\partial \chi_\alpha} + \rho_\alpha,$$

relates internal and external forces and expresses the equilibrium requirements of the element. X_α and Q_i are *statically admissible* if this equation is satisfied for some ρ_α.

Consider a set of internal variables $\hat{\chi}_\alpha$ such that $f^p(\chi_\alpha + \hat{\chi}_\alpha) = f^p(\chi_\alpha)$ for all χ_α. It follows that

$$\frac{\partial f^p}{\partial \chi_\alpha} \hat{\chi}_\alpha = -\rho_\alpha\hat{\chi}_\alpha = 0, \quad \text{for all } \chi_\alpha \tag{8a}$$

$$\frac{\partial^2 f^p}{\partial \chi_\alpha \partial \chi_\beta} \hat{\chi}_\alpha = 0, \quad \text{for all } \chi_\alpha \tag{8b}$$

$$\frac{\partial f^p}{\partial \hat{\chi}_\alpha} = 0. \tag{8c}$$

Such a set of internal variables does not change the free energy of the element and does not cause internal forces. It may therefore be termed *kinematically admissible* with deflections

$$\hat{q}_i^p = \frac{\partial q_i^p}{\partial \chi_\alpha} \hat{\chi}_\alpha. \quad (9)$$

The virtual work relation emerges as follows. Let X_α be statically admissible with Q_i and let $\hat{\chi}_\alpha$ be kinematically admissible with $\hat{q}_i^p$. From eqns (1c) and (8),

$$Q_i \hat{q}_i^p = X_\alpha \hat{\chi}_\alpha \quad (10)$$

if

$$\rho_\alpha \hat{\chi}_\alpha = 0.$$

But this is so from eqn (8a). Equation (10) is the virtual work equation.

As shown in eqn (8b), the array of constants $[\partial^2 f^p / \partial \chi_\alpha \partial \chi_\beta]$ is singular. In this case, the residual forces ρ_α span a space of dimension less than that of χ_α-space. Then the expression $-\partial j^p / \partial \rho_\alpha$ gives χ_α plus an arbitrary kinematically admissible $\hat{\chi}_\alpha$, that is,

$$\chi_\alpha = -\frac{\partial j^p}{\partial \rho_\alpha} + \hat{\chi}_\alpha.$$

4. EFFECTS OF LOADING

4.1 *Constant load*

Let the element be subjected to a constant quasi-statically applied load Q_i for time $t \geq 0$. If a limit surface ϕ_y exists, it is assumed that $\Phi(Q_i) \leq \phi_y$. By definition, the internal forces X_α are the *steady-state* internal forces X_α^{ss} if $\dot{X}_\alpha = 0$. ρ_α^{ss} are the corresponding residual forces, and $\dot{\chi}_\alpha^{ss}$ are the corresponding internal variable rates.

Theorem 1

If $X_\alpha = X_\alpha^{ss}$, then $\dot{\chi}_\alpha = \dot{\chi}_\alpha^{ss}$ is kinematically admissible.

Proof

If $X_\alpha = X_\alpha^{ss}$, then $\rho_\alpha = \rho_\alpha^{ss}$ and $\dot{\rho}_\alpha^{ss} = 0$. It follows that

$$\frac{d}{dt} j^p(\rho_\alpha^{ss}) = 0.$$

Since $f^p = j^p$, it follows that

$$\frac{\partial f^p}{\partial \chi_\alpha} \dot{\chi}_\alpha^{ss} = -\rho_\alpha^{ss} \dot{\chi}_\alpha^{ss} = 0. \quad (11)$$

For quadratic f^p, $\rho_\alpha \dot{\chi}_\alpha^{ss} = 0$ for any ρ_α. Therefore $\dot{\chi}_\alpha = \dot{\chi}_\alpha^{ss}$ is kinematically admissible [eqn (8a)].

Theorem 2

X_α^{ss} satisfies a minimum principle, namely, $\Omega(X_\alpha^{ss}) \leq \Omega(X_\alpha)$ for all statically admissible X_α.

Proof

From the convexity of Ω [eqn (3a)],

$$\Omega(X_\alpha) - \Omega(X_\alpha^{ss}) \geq \frac{\partial \Omega}{\partial X_\alpha^{ss}} (X_\alpha - X_\alpha^{ss})$$
$$= \dot{\chi}_\alpha^{ss} (\rho_\alpha - \rho_\alpha^{ss}). \quad (12)$$

But from Theorem 1, $\dot{\chi}_\alpha^{ss}$ is kinematically admissible. Therefore, the RHS of inequality (12) is zero, proving the theorem.

Theorem 3

ρ_α approaches ρ_α^{ss} monotonically, that is

$$\frac{d}{dt} j^p(\rho_\alpha - \rho_\alpha^{ss}) \begin{cases} < 0, & \text{if } \rho_\alpha \neq \rho_\alpha^{ss} \\ = 0, & \text{if } \rho_\alpha = \rho_\alpha^{ss}. \end{cases} \tag{13}$$

Proof

$$\begin{aligned} \frac{d}{dt} j^p(\rho_\alpha - \rho_\alpha^{ss}) &= \frac{d}{dt} f^p(\chi_\alpha - \chi_\alpha^{ss}) \\ &= -(\rho_\alpha - \rho_\alpha^{ss})(\dot{\chi}_\alpha - \dot{\chi}_\alpha^{ss}) \\ &= -(X_\alpha - X_\alpha^{ss})(\dot{\chi}_\alpha - \dot{\chi}_\alpha^{ss}) \\ &< 0, \quad \text{if } X_\alpha \neq X_\alpha^{ss} \\ &= 0, \quad \text{if } X_\alpha = X_\alpha^{ss} \end{aligned}$$

since Ω is strictly convex [eqn (3b)]. Since $j^p(\rho_\alpha - \rho_\alpha^{ss}) = 0$ if and only if $\rho_\alpha = \rho_\alpha^{ss}$, it follows that $\rho_\alpha \to \rho_\alpha^{ss}$.

Note that it is not essential to consider $\rho_\alpha - \rho_\alpha^{ss}$. Without any alteration, the theorem applies to, for example, $\rho_\alpha(t) - \rho_\alpha(t + \Delta t)$.

4.2 *Cyclic loading*

Let the element be subjected to a cyclic load Q_i of period T, that is, $Q_i(t) = Q_i(t + T)$. As before, if a limit surface exists, Q_i does not violate the limit criterion. The element may also be subjected to internal thermal cycling of the same period. Following Ponter[12] and Ainsworth[13], we define three kinds of solution, that is, histories of X_α and ρ_α.

(i) The *steady cyclic solution* X_α^{sc} satisfies $X_\alpha^{sc}(t) = X_\alpha^{sc}(t + T)$. ρ_α^{sc} is the corresponding residual force history, and $\dot{\chi}_\alpha^{sc}$ is the corresponding internal variable rate.

(ii) The *cyclic plasticity solution* X_α^{cp}, ρ_α^{cp}, $\dot{\chi}_\alpha^{cp}$ is the cyclic solution obtained if all time-dependent behaviour is suppressed [$k = 0$ in eqns (4)]. ρ_α^{cp} is uniquely defined if cyclic plasticity occurs, whereas $\dot{\rho}_\alpha^{cp}$ is uniquely defined if shakedown occurs[13].

(iii) The *rapid cycle solution* X_α^{rc}, ρ_α^{rc} is the cyclic plasticity solution for which

$$\int_{\text{cycle}} \dot{\chi}_\alpha \, dt = \int_{\text{cycle}} \dot{\bar{\chi}}_\alpha \, dt \tag{14}$$

is kinematically admissible, where $\dot{\bar{\chi}}_\alpha = \partial\Omega/\partial X_\alpha^{cp}$ are the full kinetic equations. Then we may define $\dot{\bar{\chi}}_\alpha^{rc} = \partial\Omega/\partial X_\alpha^{rc}$.

Theorem 4

If $X_\alpha = X_\alpha^{sc}$, then $\Delta\chi_\alpha^{sc} = \int_{\text{cycle}} \dot{\bar{\chi}}_\alpha^{sc} \, dt$ is kinematically admissible.

Proof

Since ρ_α is cyclic, $j^p(\rho_\alpha)$ and hence $f^p(\chi_\alpha)$ are cyclic. Therefore,

$$f^p[\chi_\alpha(t)] = f^p[\chi_\alpha(t + T)] = f^p[\chi_\alpha(t) + \Delta\chi_\alpha^{sc}]. \tag{15}$$

By definition (Section 3), $\Delta\chi_\alpha^{sc}$ is kinematically admissible.

Theorem 5

X_α^{sc} satisfies a minimum principle, namely,

$$\int_{\text{cycle}} \Omega(X_\alpha^{sc})\,dt \leq \int_{\text{cycle}} \Omega(X_\alpha^{cp})\,dt \tag{16}$$

for any cyclic plasticity solution X_α^{cp}.

Proof

From the convexity of Ω,

$$\int_{\text{cycle}} \Omega(X_\alpha^{cp})\,dt - \int_{\text{cycle}} \Omega(X_\alpha^{sc})\,dt \geq \int_{\text{cycle}} \frac{\partial\Omega}{\partial X_\alpha^{sc}}(X_\alpha^{cp} - X_\alpha^{sc})\,dt$$

$$= -\int_{\text{cycle}} (\dot{\chi}_\alpha^{cp} - \dot{\chi}_\alpha^{sc})(\rho_\alpha^{cp} - \rho_\alpha^{sc})\,dt + \int_{\text{cycle}} \dot{\chi}_\alpha^{cp}(X_\alpha^{cp} - X_\alpha^{sc})\,dt. \tag{17}$$

The first term in this inequality is zero, since its integrand is equal to

$$\frac{d}{dt} j^p(\rho_\alpha^{cp} - \rho_\alpha^{sc}),$$

the argument of which is cyclic. As noted in Section 2.2, the convexity of $D(\dot{\chi}_\alpha)$ for the time-independent case implies the maximum plastic work inequality $(X_\alpha - X_\alpha^*)\dot{\chi}_\alpha \geq 0$. The second term is therefore nonnegative, proving the theorem.

Theorem 6

Of all the cyclic plasticity solutions X_α^{cp}, the rapid cycle solution X_α^{rc} satisfies a minimum principle, namely,

$$\int_{\text{cycle}} \Omega(X_\alpha^{rc})\,dt \leq \int_{\text{cycle}} \Omega(X_\alpha^{cp})\,dt. \tag{18}$$

Proof

From the convexity of Ω,

$$\int_{\text{cycle}} \Omega(X_\alpha^{cp})\,dt - \int_{\text{cycle}} \Omega(X_\alpha^{rc})\,dt \geq \int_{\text{cycle}} \dot{\chi}_\alpha^{rc}(X_\alpha^{cp} - X_\alpha^{rc})\,dt$$

$$= \int_{\text{cycle}} \dot{\chi}_\alpha^{rc}(\rho_\alpha^{cp} - \rho_\alpha^{rc})\,dt. \tag{19}$$

We have noted that the cyclic plasticity solution is uniquely defined (to within an arbitrary constant residual force if shakedown occurs). Therefore, $\rho_\alpha^{cp} - \rho_\alpha^{rc}$ is constant in time, and the right hand side of inequality (19) is zero since

$$\Delta\chi_\alpha^{rc} = \int_{\text{cycle}} \dot{\chi}_\alpha^{rc}\,dt$$

is by definition kinematically admissible. This proves the theorem.

Theorem 7

X_α approaches X_α^{sc} monotonically, that is

$$\frac{d}{dt} j^p(\rho_\alpha - \rho_\alpha^{sc}) \begin{cases} < 0, & \text{if } \rho_\alpha \neq \rho_\alpha^{sc} \\ = 0, & \text{if } \rho_\alpha = \rho_\alpha^{sc}. \end{cases}$$

Proof

$$\frac{d}{dt}j^p(\rho_\alpha - \rho_\alpha^{sc}) = \frac{d}{dt}f^p(\chi_\alpha - \chi_\alpha^{sc})$$
$$= -(\rho_\alpha - \rho_\alpha^{sc})(\dot{\chi}_\alpha - \dot{\chi}_\alpha^{sc})$$
$$= -(X_\alpha - X_\alpha^{sc})(\dot{\bar{\chi}}_\alpha - \dot{\bar{\chi}}_\alpha^{sc})$$
$$< 0, \quad \text{if } X_\alpha \neq X_\alpha^{sc}$$
$$= 0, \quad \text{if } X_\alpha = X_\alpha^{sc},$$

since Ω is strictly convex [eqn (3b)]. Since $j^p(\rho_\alpha - \rho_\alpha^{sc}) = 0$ if and only if $\rho_\alpha = \rho_\alpha^{sc}$, it follows that $\rho_\alpha \rightarrow \rho_\alpha^{sc}$.

As in Theorem 3, we note that it is not essential to consider $\rho_\alpha - \rho_\alpha^{sc}$. The theorem also applies to, say, $X_\alpha(t) - X_\alpha(t + T)$.

Theorems 5 and 6 may be summarised by

$$\int_{\text{cycle}} \Omega(X_\alpha^{sc})\, dt \leq \int_{\text{cycle}} \Omega(X_\alpha^{rc})\, dt \leq \int_{\text{cycle}} \Omega(X_\alpha^{cp})\, dt. \tag{20}$$

In the case of n-power creep, Ω in these theorems is replaced by the energy dissipation rate $D(X_\alpha)$. We see that there is no general physical requirement that $D(X_\alpha)$ be minimised for steady and cyclic loading. It is, however, possible to calculate work and displacement bounds using a technique developed by Ponter.

5. BOUNDING PROPERTIES

In [12,21] and previous articles, Ponter derived displacement and work bounds on inelastic bodies subject to quasi-static time-varying loads. Subsequently[22], inertial effects were included to give general bounding theorems, which include as special cases the previous results and the results of Martin[23] for the maximum displacement at a point on an elastic–plastic body subject to an impulse. In the theory, a functional W was defined as follows: Let $Q_i^*(t)$ $(0 \leq t \leq T)$ be a prescribed load history and let $Q_i(t)$ be an independent load history with its associated inelastic displacement rate history $\dot{q}_i^p$. Then,

$$W = \int_0^T (Q_i^* - Q_i)\dot{q}_i^p\, dt.$$

Central to the theory is the assumption that for any $Q_i^*(t)$ there exists $w(Q_i^*)$ such that $W \leq w$ for all histories $Q_i(t)$ $(0 \leq t \leq T)$.

5.1 *Case 1:* Q_i^* = *constant*

Martin[23] and Ponter[24] obtained the following inequality for time-dependent and time-independent materials:

$$\int_0^{q_i} Q_i\, dq_i + U(Q_i^*) \leq Q_i^* q_i, \tag{21}$$

where

$$U(Q_i^*) = \text{maximum complementary work to } Q_i^*$$
$$= \max.\left[\int_0^{Q_i^*} q_i\, dQ_i : q_i|_{Q_i=0} = 0\right].$$

Inequality (21) may be rearranged to give

$$\int_0^{q_i} (Q_i^* - Q_i)\,\mathrm{d}q_i \leq U(Q_i^*).$$

In this case, then, $w(Q_i^*)$ = maximum complementary work to Q_i^*.

Carter and Martin[7] have obtained the conditions for extremal or m-paths for a material model similar to that of this article. If an m-path is followed, then

$$\int_0^{Q_i^*} q_i\,\mathrm{d}Q_i = U(Q_i^*).$$

However, it appears that the concept is of limited use, since it does not seem possible to calculate $U(Q_i^*)$ explicitly in the general case $f^p \neq 0$. In this section, we adopt an alternative approach to bounding W.

As described in Section 2, we consider a material having $j = h^e(Q_i) + j^p(\rho_\alpha)$, $\dot{\chi}_\alpha = \partial\Omega/\partial X_\alpha$, where h^e, j^p and Ω are convex. Let X_α^* be constant internal forces in equilibrium with Q_i^*, that is,

$$X_\alpha^* = Q_i^* \frac{\partial q_i^p}{\partial \chi_\alpha} + \rho_\alpha^*,$$

where ρ_α^* values are arbitrary constant residual forces. Then using $q_i^p = (\partial q_i^p/\partial\chi_\alpha)\,\chi_\alpha$,

$$\begin{aligned} W &= \int_0^T (Q_i^* - Q_i)\dot{q}_i^p\,\mathrm{d}t \\ &= \int_0^T (X_\alpha^* - X_\alpha)\dot{\chi}_\alpha\,\mathrm{d}t - \int_0^T (\rho_\alpha^* - \rho_\alpha)\dot{\chi}_\alpha\,\mathrm{d}t. \end{aligned}$$

Bounds are obtained for the two terms in this equation separately. The first term is bounded by considering two paths, $\chi_\alpha'(t)$ and $\chi_\alpha''(t)$. Let this term be denoted W' and W'' for these paths. Then,

$$W'' - W' = \int_0^T [X_\alpha^*(\dot{\chi}_\alpha'' - \dot{\chi}_\alpha') - D(\dot{\chi}_\alpha'') + D(\dot{\chi}_\alpha')]\,\mathrm{d}t.$$

But since $D(\dot{\chi}_\alpha)$ is convex,

$$D(\dot{\chi}_\alpha'') - D(\dot{\chi}_\alpha') \geq \frac{\partial D}{\partial \dot{\chi}_\alpha'}(\dot{\chi}_\alpha'' - \dot{\chi}_\alpha').$$

Therefore, $W' \leq W''$ if

$$\frac{\partial D}{\partial \dot{\chi}_\alpha'} = X_\alpha^*, \qquad 0 \leq t \leq T. \tag{22}$$

Since X_α^* is constant, this is satisfied if $\dot{\chi}_\alpha'$ = constant = $\hat{\chi}_\alpha/T$, say. Then,

$$\int_0^T (X_\alpha^* - X_\alpha)\dot{\chi}_\alpha\,\mathrm{d}t \leq X_\alpha^*\hat{\chi}_\alpha - TD\left(\frac{\hat{\chi}_\alpha}{T}\right),$$

where $\dot{\chi}_\alpha = \hat{\chi}_\alpha/T$ satisfies eqn (22).

The second term to be bounded is

$$-\int_0^T (\rho_\alpha^* - \rho_\alpha)\dot{\chi}_\alpha\,\mathrm{d}t = -\rho_\alpha^*\hat{\chi}_\alpha - f^p(\hat{\chi}_\alpha), \tag{23}$$

since ρ_α^* is constant and $\rho_\alpha = -\partial f^p/\partial\chi_\alpha$. If we consider variations in $\hat{\chi}_\alpha$ and use the convexity of f^p, it is also readily shown that the second term is minimised if

$$\rho_\alpha^* = -\frac{\partial f^p}{\partial \chi_\alpha}.$$

Since $j^p = (\partial f^p/\partial\chi_\alpha)\chi_\alpha - f^p$ [eqn (2b)], it follows that

$$-\int_0^T (\rho_\alpha^* - \rho_\alpha)\dot{\chi}_\alpha \, dt \leq j^p(\rho_\alpha^*).$$

We have shown that W is bounded for convex j^p and $D(\dot{\chi}_\alpha)$. To calculate $w(Q_i^*)$ explicitly, consider the creep-plasticity model described in eqns (4). In this case, from eqn (6),

$$\frac{\partial D}{\partial \dot{\chi}_\alpha} = \begin{cases} \dfrac{n+1}{n} X_\alpha, & \text{if } \phi \leq \dfrac{n}{n+1}\phi_y \\[2ex] \dfrac{\phi_y}{\phi(X_\alpha)} X_\alpha, & \text{if } \dfrac{n}{n+1}\phi_y \leq \phi \leq \phi_y. \end{cases} \tag{24}$$

Equation (22) for a minimum path becomes

$$X_\alpha(t) = \frac{n}{n+1} X_\alpha^*, \quad \text{for } \phi \leq \phi_y.$$

Note that if $\phi(X_\alpha^*) = \phi_y$, then eqn (22) is satisfied by any $X_\alpha = \lambda X_\alpha^*$ with $[n/(n+1)]\phi_y \leq \phi \leq \phi_y$. Using eqns (4) for these paths gives

$$\int_0^T (X_\alpha^* - X_\alpha)\dot{\chi}_\alpha \, dt \leq T\frac{k}{n}\phi^{n+1}\left(\frac{n}{n+1}X_\alpha^*\right).$$

Therefore,

$$W = \int_0^T (Q_i^* - Q_i)\dot{q}_i^p \, dt \leq w, \tag{25a}$$

where

$$w(Q_i^*) = T\frac{k}{n}\phi^{n+1}\left(\frac{n}{n+1}X_\alpha^*\right) + j^p(\rho_\alpha^*) \tag{25b}$$

$$X_\alpha^* = Q_i^*\frac{\partial q_i^p}{\partial \chi_\alpha} + \rho_\alpha^*. \tag{25c}$$

5.1.1 *Work bound.* As in Section 5.1, let the element be subjected to a constant quasi-statically applied load Q_i for $t \geq 0$. Consider two cases.

Case 1. If $\Phi(Q_i) \leq [n/(n+1)]\phi_y$, following Ponter[12], choose

$$Q_i^* = \frac{n+1}{n} Q_i$$

$$X_\alpha^* = \frac{n+1}{n} X_\alpha^g,$$

that is, X_α^g values are any internal forces in equilibrium with Q_i, or

$$X_\alpha^g = Q_i \frac{\partial q_i^p}{\partial \chi_\alpha} + \rho_\alpha^g.$$

Inequality (25) then becomes

$$\int_0^T Q_i \dot{q}_i^p \, dt \leqslant n \cdot j^p \left(\frac{n+1}{n} \rho_\alpha^g\right) + Tk\phi^{n+1}(X_\alpha^g). \tag{26}$$

A less conservative work bound for large times is obtained by putting $X_\alpha^g = X_\alpha^{ss}$ and $\rho_\alpha^g = \rho_\alpha^{ss}$. Equation (23) may be calculated for $\rho_\alpha^* = [(n+1)/n]\rho_\alpha^{ss}$ to give

$$\int_0^T Q_i \dot{q}_i^p \leqslant (n+2) j^p(\rho_\alpha^{ss}) + Tk\phi^{n+1}(X_\alpha^{ss}).$$

Case 2. If $[n/(n+1)]\phi_y \leqslant \Phi(Q_i) \leqslant \phi_y$, choose $Q_i^* = \lambda Q_i$ such that $\Phi(Q_i^*) = \phi_y$. Now, $\Phi(Q_i)$ is homogeneous and of degree 1 in Q_i, and it may be shown that

$$\begin{aligned} Q_i^* - Q_i &= \frac{Q_i[\phi_y - \Phi(Q_i)]}{\Phi(Q_i)} \\ &\geqslant \frac{Q_i[\phi_y - \phi(X_\alpha^g)]}{\phi(X_\alpha^g)} \end{aligned}$$

for any X_α^g in equilibrium with Q_i, where use is made of inequality (7). Inequality (25) then becomes

$$\int_0^T Q_i \dot{q}_i^p \, dt \leqslant \frac{\phi(X_\alpha^g)}{\phi_y - \phi(X_\alpha^g)} \left[j^p(\rho_\alpha^*) + T\frac{k}{n}\left(\frac{n}{n+1}\phi_y\right)^{n+1} \right]. \tag{27}$$

Here ρ_α^* values are residual forces such that $\phi(Q_i, \rho_\alpha^*) \leqslant \phi(Q_i, \rho_\alpha)$ as in inequality (7). Examining inequalities (26) and (27), the rate of work at large times may be bounded as follows:

$$Q_i \dot{q}_i^p \leqslant \begin{cases} k\phi^{n+1}(X_\alpha^g), & \text{if } \phi \leqslant \left(\dfrac{n}{n+1}\right)\phi_y \\ \dfrac{\phi(X_\alpha^g)}{\phi_y - \phi(X_\alpha^g)} \dfrac{k}{n}\left(\dfrac{n}{n+1}\phi_y\right)^{n+1}, & \text{if } \left(\dfrac{n}{n+1}\right)\phi_y \leqslant \Phi \leqslant \phi_y. \end{cases}$$

Examining eqns (5), we see that at large times

$$Q_i \dot{q}_i^p = D(X_\alpha^{ss}) \leqslant D(X_\alpha^g) \tag{28}$$

for all safe statically admissible X_α^g.

5.2 *Case 2: Cyclic loading*

For this case, the method of Ainsworth[13] is adapted as follows. Consider two geometrically identical elements subject to cyclic load histories $Q_i(t)$ and $Q_i^*(t)$ of period T. They are also subject to identical thermal histories that induce identical thermal components $\chi_\alpha^T(t)$ of the internal variables. At this stage, we identify the unstarred material with the creep-plasticity model discussed in Section 2.2. The starred material is identical except that the time-dependent behaviour is suppressed [$k = 0$ in eqn (4)].

The corresponding inelastic deformation rates are $\dot{q}_i^p$ and $\dot{q}_i^{*p}$. It will be assumed that both elements have attained their steady cyclic states. We seek a lower bound to

$$\overline{W} = \int_{\text{cycle}} (Q_i^* - Q_i)(\dot{q}_i^p - \dot{q}_i^{*p})\,\mathrm{d}t.$$

Using the equilibrium equation (1c) for each element and $q_i^p = (\partial q_i^p/\partial \chi_\alpha)\chi_\alpha$,

$$\overline{W} = \int_{\text{cycle}} (X_\alpha^* - X_\alpha)\dot{\overline{\chi}}_\alpha\,\mathrm{d}t + \int_{\text{cycle}} (X_\alpha - X_\alpha^*)\dot{\overline{\chi}}_\alpha^*\,\mathrm{d}t$$
$$+ \int_{\text{cycle}} (\rho_\alpha^* - \rho_\alpha)(\dot{\chi}_\alpha^* - \dot{\chi}_\alpha)\,\mathrm{d}t.$$

The last term in this equation is zero since its integrand is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t} j^p(\rho_\alpha^* - \rho_\alpha),$$

the argument of which is cyclic. As noted in Section 2.2, the convexity of $D(\dot{\chi}_\alpha^*)$ implies the maximum plastic work inequality $(X_\alpha^* - X_\alpha)\dot{\overline{\chi}}_\alpha^* \geq 0$. The second term may therefore be ignored. The first term may be bounded using the procedure of Section 5.1. The requirement for a minimum path is given by eqn (22), namely,

$$\frac{\partial D}{\partial \dot{\overline{\chi}}_\alpha} = X_\alpha^*.$$

Again, using the same arguments as in Section 5.1 and eqns (24), it can be seen that

$$\overline{W} \leq \int_{\text{cycle}} (X_\alpha^* - X_\alpha)\dot{\overline{\chi}}_\alpha\,\mathrm{d}t \leq \int_{\text{cycle}} \frac{k}{n}\phi^{n+1}\left(\frac{n}{n+1}X_\alpha^*\right)\mathrm{d}t.$$

Therefore,

$$\int_{\text{cycle}} (Q_i^* - Q_i)\dot{q}_i^p\,\mathrm{d}t \leq \int_{\text{cycle}} (Q_i^* - Q_i)\dot{q}_i^{*p}\,\mathrm{d}t + \int_{\text{cycle}} \frac{k}{n}\phi^{n+1}\left(\frac{n}{n+1}X_\alpha^*\right)\mathrm{d}t. \quad (29)$$

This inequality may be used to obtain displacement and work bounds by various choices of Q_i^*.

5.2.1 *Displacement and work bounds.* Following Ainsworth[13], we choose $Q_i^* = Q_i + R_i$, where R_i is constant in time. Inequality (29) becomes

$$\Delta q_R = \Delta q_R^p \leq \Delta q_R^{*p} + \frac{1}{n\,|R|}\int_{\text{cycle}} D\left(\frac{n}{n+1}X_\alpha^*\right)\mathrm{d}t, \quad (30)$$

where Δq_R is the increment in q_i over a cycle in the direction of R_i (similarly for Δq_R^p, Δq_R^{*p}) and $|R| = (R_iR_i)^{1/2}$.

Choose $Q_i^* = \lambda Q_i$ where λ is a constant. Inequality (29) becomes

$$\int_{\text{cycle}} Q_i\dot{q}_i^p\,\mathrm{d}t \leq \int_{\text{cycle}} Q_i\dot{q}_i^{*p}\,\mathrm{d}t + \frac{1}{n(\lambda - 1)}\int_{\text{cycle}} D\left(\frac{n}{n+1}X_\alpha^*\right)\mathrm{dt}. \quad (31)$$

This gives an upper bound to the work done by the boundary forces Q_i over one cycle.

A more meaningful bound may be obtained in certain cases. First we attempt to choose (as in [12])

$$Q_i^* = \frac{n+1}{n} Q_i$$

such that

$$\dot{\chi}_\alpha^{*cp} = 0,$$

that is X_α^{*cp} is a shakedown solution. Then,

$$X_\alpha^{*cp} = Q_i^* \frac{\partial q_i^p}{\partial \chi_\alpha} + \rho_\alpha^T + \rho_\alpha,$$

where ρ_α^T is the elastic thermal residual force history induced by χ_α^T and ρ_α is a constant. Define

$$\begin{aligned} X_\alpha^g &= \frac{n}{n+1} X_\alpha^{*cp} \\ &= Q_i \frac{\partial q_i^p}{\partial \chi_\alpha} + \frac{n}{n+1} (\rho_\alpha^T + \rho_\alpha), \end{aligned}$$

which is the shakedown solution for mechanical loading Q_i and thermal loading $[n/(n+1)]\rho_\alpha^T$. With $\lambda = (n+1)/n$, inequality (31) becomes

$$\int Q_i \dot{q}_i^p \, \mathrm{d}t \le \int D(X_\alpha^g) \, \mathrm{d}t. \tag{32}$$

If this is not possible over the complete cycle, choose Q_i^* in this way over only those parts of the cycle where $\dot{\chi}_\alpha^{*cp} = 0$. For the rest of the cycle, choose $Q_i^* = \lambda Q_i$, where λ is the largest positive constant such that X_α^* shakes down. Choose $X_\alpha^g = X_\alpha^{*cp}/\lambda$. Then, arguing similarly to the previous case, X_α^g is a shakedown solution for mechanical loading Q_i and thermal loading ρ_α^T/λ. Since $\phi(X_\alpha)$ is homogeneous and of degree 1 in X_α and $\phi(X_\alpha^*) \le \phi_y$,

$$\lambda - 1 \le \frac{\phi_y - \phi(X_\alpha^g)}{\phi(X_\alpha^g)}.$$

Inequality (29) becomes

$$\begin{aligned} \int Q_i \dot{q}_i^p \, \mathrm{d}t &\le \int \frac{\phi(X_\alpha^g)}{\phi_y - \phi(X_\alpha^g)} \frac{k}{n} \left(\frac{n}{n+1} \phi_y \right)^{n+1} \mathrm{d}t \\ &= \int D(X_\alpha^g) \, \mathrm{d}t \end{aligned} \tag{33}$$

using eqn (5). From eqns (32) and (33),

$$\int_{\text{cycle}} Q_i \dot{q}_i^p \, \mathrm{d}t = \int_{\text{cycle}} D(X_\alpha^{sc}) \, \mathrm{d}t \le \int_{\text{cycle}} D(X_\alpha^g) \, \mathrm{d}t, \tag{34}$$

where X_α^g is any shakedown solution for mechanical loading Q_i and thermal loading ρ_α^T/λ, where $\lambda = (n+1)/n$ when $\phi(X_\alpha^g) \le [n/(n+1)]\phi_y$. Otherwise,

$$\lambda = \text{min.} \left\{ \frac{\phi_y}{\phi(X_\alpha^g)} : 1 \le \lambda \le \frac{n+1}{n} \right\}.$$

6. CONCLUSION

As well as exploring the role of convexity in a number of bounding theorems, we have extended the use of reference stress methods for high-temperature design beyond the modified $[n/(n + 1)]$ yield and shakedown limits (see, for example, [21] and [25]). This complements the work of Ponter[26] who argued that the $n/(n + 1)$ limit should not apply to materials whose steady-state creep rate is unaffected by the initial yield stress, that is, when the stress–strain rate relation is continuous up to the high-temperature ultimate stress. Here, we have shown that the $n/(n + 1)$ limit is not even an essential feature of materials that *are* reasonably modelled by perfect plasticity and n-power creep.

The consequences for high-temperature design will be discussed in a future paper. It may be noted here that reference stress calculations in which one uses arbitrary statically admissible stress distributions to bound D are particularly powerful when thermal stresses are low compared with primary or load-induced stresses. If they are not, the work bound tends to be too conservative to be useful. A better approach in this case is to calculate the strain increments associated with the rapid cycle solution. This has been done using plane constant strain elements by Ponter and Brown[27] for creep and by O'Donnell and Porowski[28] for creep-plasticity in the Bree nuclear fuel can problem.

REFERENCES

1. B. D. Coleman and W. Noll, The thermodynamics of elastic materials with heat conduction and viscosity. *Arch. Rat. Mech. Anal.* **13**, 167–178 (1963).
2. J. Kestin and J. R. Rice, Paradoxes in the application of thermodynamics to strained solids. In *A Critical Review of Thermodynamics* (Edited by E. B. Stuart, B. Gal-or and A. J. Brainard), pp. 275–296. Mono Book Corp., New York (1970).
3. J. B. Martin, *Plasticity, Part VI.* MIT Press, Cambridge (1975).
4. J. H. Lambermont, On the unification of the mathematical theory of plasticity and dislocation theory using thermodynamics, I. Basic theory. *Int. J. Engng Sci.* **12**, 937–952 (1974).
5. J. H. Lambermont, On the unification of the mathematical theory of plasticity and dislocation theory using thermodynamics, II. *Int. J. Engng Sci.* **12**, 953–965 (1974).
6. A. R. S. Ponter, J. Bataille and J. Kestin, A thermodynamic model for the time dependent plastic deformation of solids. *J. Mécanique* **18**, 511–539 (1979).
7. P. Carter and J. B. Martin, Work bounding functions for plastic materials. *J. Appl. Mech. Trans ASME*, **43**, 434–448 (1976).
8. P. Carter and J. B. Martin, A note on the derivation of the minimum rate theorem of plasticity from the free energy minimum principle. *J. Mécanique* **16**, 791–802 (1977).
9. P. Carter and J. B. Martin, The derivation of the static rate theorems of plasticity from an internal variable model. *J. Mech. Phys. Solids* **26**, 345–353 (1979).
10. F. A. Leckie, A review of bounding techniques in shakedown and ratchetting at elevated temperatures. *Welding Res. Council Bull.* **195**, 1–32 (1974).
11. D. L. Marriott, A review of reference stress methods for estimating creep deformation. *Proc. IUTAM Symposium on the Creep of Structures, Gothenburg*, pp. 137–152 (1970).
12. A. R. S. Ponter, Deformation, displacement and work bounds for structures in a state of creep and subject to variable loading. *J. Appl. Mech.* **39**, 953–958 (1972).
13. R. A. Ainsworth, Bounding solutions for creeping structures subject to load variations above the shakedown limit. *Int. J. Solids Structures* **13**, 971–980 (1977).
14. O. D. Sherby and A. K. Miller, Combining phenomenology and physics in describing the high temperature mechanical behaviour of crystalline solids. *ASME J. Engng Materials Technol.* **101**, 387–394 (1979).
15. E. W. Hart, A phenomenological theory for plastic deformation of polycrystalline metals. *Acta Metall.* **18**, 599 (1970).
16. J. Kratochvil, Meaning of internal variables in plasticity. *Lett. Appl. Engng Sci.* **16**, 403–414 (1978).
17. P. Carter, An internal variable approach to elastic-plastic fracture mechanics. *Engng Fracture Mech.* **11**, 441–456 (1979).
18. H. B. Callen, *Thermodynamics.* Wiley, New York (1960).
19. J. B. Martin, A note on the implications of thermodynamic stability in the internal variable theory of inelastic solids. *Int. J. Solids Structures* **11**, 247–253 (1975).
20. D. C. Drucker, A more fundamental approach to plastic stress–strain relations. *Proc. 1st U.S. Nat. Cong. Appl. Mech., ASME*, p. 487 (1951).
21. A. R. S. Ponter, The derivation of energy theorems for creep constitutive relationships. *Proc. IUTAM Symposium on the Creep of Structures, Gothenburg*, p. 123 (1970).
22. A. R. S. Ponter, General displacement and work bounds for dynamically loaded bodies. *J. Mech. Phys. Solids* **23**, 151–163 (1975).
23. J. B. Martin, Extended displacement bound theorems for work hardening continua subjected to dynamic loading. *Int. J. Solids Structures* **2**, 9–26 (1966).

24. A. R. S. Ponter, An energy theorem for time dependent materials. *J. Mech. Phys. Solids* **17**, 63–71 (1969).
25. R. A. Ainsworth and I. W. Goodall, Proposals for primary design above the creep threshold temperature. CEGB Report RD/B/N4394 (1978).
26. A. R. S. Ponter, On the creep modified shakedown limit. In *Proc. IUTAM Symposium on Creep in Structures, Leicester* (Edited by A. R. S. Ponter and D. R. Hayhurst), pp. 264–278. Springer-Verlag, New York (1980).
27. A. R. S. Ponter and P. R. Brown, The finite element solutions of rapid cycling creep problems. *Int. J. Numerical Methods Engng* **2**, 1001–1024 (1978).
28. W. J. O'Donnell and J. Porowski, Upper bounds for accumulated strains due to creep ratchetting. *ASME J. Press. Vess. Tech.* **96**, 150 (1974).